MATH 2850: 5.2 - CONSTANT COEFFICIENT HOMOGENEOUS SECOND ORDER ODEs

RECALL: $D_x\left[e^{mx}\right] = m\,e^{mx}$. That is, the derivatives of basic exponential functions are multiples of themselves.

EXAMPLE: Let $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$. Prove if $m_1 \neq m_2$, then y_1 and y_2 are linearly independent.

Consider the so-called 'constant coefficient' DE $a_2 y'' + a_1 y' + a_0 y = 0$.

We are looking for a function y so that a linear combination of it and its derivatives are 0. So we try $y = e^{mx}$.

Substituting $y = e^{mx}$ into the DE gives: $a_2 m^2 e^{mx} + a_1 m e^{mx} + a_0 e^{mx} = 0$.

Factoring, we get $e^{mx}\left[a_2 m^2 + a_1 m + a_0\right] = 0$. Since $e^{mx} \neq 0$, we'd need: $a_2 m^2 + a_1 m + a_0 = 0$.

The quadratic equation $a_2 m^2 + a_1 m + a_0 = 0$ is called the 'auxiliary' or 'ancillary' equation.

If $a_2 m^2 + a_1 m + a_0 = 0$ has distinct solutions, we'll have two linearly independent solutions to the DE!

EXAMPLE: Solve the following DE's:

- $2y'' - 5y' - 3y = 0$

Ans: $y = c_1 e^{-\frac{1}{2}x} + c_2 e^{3x}$

- $y'' = 2y' + y$

Ans: $y = c_1 e^{(1-\sqrt{2})x} + c_2 e^{(1+\sqrt{2})x}$

EXAMPLE: Solve the DE: $y'' = k^2 y$ assuming $k > 0$ is a constant.

Ans: $y = c_1 e^{-kx} + c_2 e^{kx} = c_1 \sinh(kx) + c_2 \cosh(kx)$

EXAMPLE: What happens when you try to solve $y'' - 6y' + 9y = 0$?

REDUCTION OF ORDER:

Given y_1 is a nontrivial solution to a DE, we try to find u so $y = u\, y_1$ is also a solution.

EXAMPLE: Use reduction of order to solve $y'' - 6y' + 9y = 0$ given $y = e^{3x}$ is a solution.

Ans: $y = c_1 e^{3x} + c_2 x e^{3x}$

EXAMPLE: Show $\{e^{mx}, xe^{mx}\}$ is a linearly independent set of functions.

THEOREM: If m is a repeated root of the auxiliary equation of $a_2 y'' + a_1 y' + a_0 y = 0$, then $y = e^{mx}$ and $y = xe^{mx}$ are two linearly independent solutions. Hence the general solution is $y = c_1 e^{mx} + c_2 x e^{mx}$.

EXAMPLE: Solve $y'' + 25y = 10y'$, $y(0) = -2$, $y'(0) = 2$

Ans: $y = 12xe^{5x} - 2e^{5x}$

EXAMPLE: What happens when you try to solve $y'' + 4y = 0$?

EULER'S FORMULA: $e^{ix} = \cos(x) + i\,\sin(x)$. In general:

$$e^{(a+bi)x} = e^{ax+ibx} = e^{ax}e^{i(bx)} = e^{ax}\cos(bx) + ie^{ax}\sin(bx)$$

EXAMPLE: Solve the following.

- $y'' + 4y = 0$

Ans: $y = c_1 \sin(2x) + c_2 \cos(2x)$

- $y'' + \omega^2 y = 0$

Ans: $y = c_1 \sin(\omega x) + c_2 \cos(\omega x)$

- $y'' + 2y' + 4y = 0$

Ans: $y = c_1 e^{-x} \sin(x\sqrt{3}) + c_2 e^{-x} \cos(x\sqrt{3})$

SUMMARY: If the solutions to the auxiliary equation for the DE $a_2y'' + a_1y' + a_0y = 0$ is/are:

- two distinct real numbers, m_1 and m_2, the general solution is: $y = c_1e^{m_1x} + c_2e^{m_2x}$
- one repeated real number, m, the general solution is: $y = c_1e^{mx} + c_2xe^{mx}$
- two purely imaginary numbers $\pm bi$, the general solution is: $y = c_1\sin(bx) + c_2\cos(bx)$
- two nonzero complex numbers $a \pm bi$, the general solution is: $y = e^{ax}\left[c_1\sin(bx) + c_2\cos(bx)\right]$

ALTERNATIVE SOLUTION FORMS: Sinusoids. Recall from Trigonometry:

$$a\sin(\omega x) + b\cos(\omega x) = \sqrt{a^2+b^2}\sin(\omega x + \phi), \quad \text{where} \quad \cos(\phi) = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin(\phi) = \frac{b}{\sqrt{a^2+b^2}}$$

Hence, we may also write the solution to: $y'' + \omega^2 y = 0$ as $y = c_1\sin(\omega x + c_2)$.

EXAMPLE: Solve $y'' + 4y = 0$, $y(0) = -3$, $y'(0) = 8$.

Write your answer in the form $y = c_1\sin(2x) + c_2\cos(2x)$ and $y = c_1\sin(\omega x + c_2)$.

Ans: $y = 5\sin\left(2x - \sin^{-1}\left(\frac{3}{5}\right)\right)$

HOMEWORK: Pg. 217: 1-21 odd, 33*, 34*